JHEP HYPER VERSION

Received: *May 31, 2001*, Accepted: *October 30, 2001*
Revised : *September 11, 2001*

A consistency check for renormalons in lattice gauge theory: β^{-10} contributions to the $\mathrm{SU}(3)$ plaquette*

Francesco Di Renzo[a] **and Luigi Scorzato**[ab]

[a]*Dipartimento di Fisica, Università di Parma and INFN, Gruppo Collegato di Parma, Italy*
[b]*Department of Physics, University of Wales Swansea Singleton Park, Swansea SA2 8PP, United Kingdom*
E-mail: direnzo@parma.infn.it, l.scorzato@swansea.ac.uk

Abstract: We compute the perturbative expansion of the Lattice SU(3) plaquette to order β^{-10} on 8^4 and 24^4 volumes. The result is found to be consistent both with the expected renormalon behaviour and with finite size effects on top of that. Complete control over the latter effects was the only loose end still to be tied up from previous works of our group.

Keywords: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories.

*Research supported by Italian MURST under contract 9702213582, by I.N.F.N. under *i.s. PR11* and by EU Grant EBR-FMRX-CT97-0122.

JHEP10(2001)038

Contents

1. Introduction

In recent years, the numerical implementation of Stochastic Perturbation Theory (NSPT) has been introduced, which was able to reach unprecedented high orders in perturbative expansions in Lattice Gauge Theory (LGT). The principal success of NSPT was the computation of the Lattice SU(3) basic plaquette to order β^{-8} [1]. This led to the possibility of actually verifying the expected dominance of the leading InfraRed (IR) Renormalon [2] associated with a dimension four condensate. The computation was performed on a 8^4 lattice. We are now in a position to quote results to a reasonable accuracy for the perturbative expansion of the same quantity to order β^{-10} on both 8^4 and 24^4 lattices.

There are at least three main motivations for such a computation. First of all, it ties up what was probably the only loose end left over from our previous work, i.e. a full assessment of finite size effects. As a matter of fact, while there is (almost) general consensus on the conjectured asymptotic dominance of renormalon contributions, to our knowledge the results in [1] are actually the only ones from which one can recognize the presence of renormalons by direct inspection of the perturbative coefficients of a QCD quantity. Ironically, this is true in a scheme (the lattice) which one would never choose for a high order QCD calculation in a standard (as opposed to NSPT) approach. Still, finite size effects have always been a matter of concern for the computation. They play a fundamental role since they impinge on the IR region which is responsible for the renormalon growth. Basically, the higher the order of the loop, the lower the (IR) momenta which give the main contribution. In view of this one could be worried that one is missing important contributions on the 8^4 lattice on which the NSPT perturbative computations were first performed: the taming of the factorial growth by the finite volume could prevent one from properly singling out the leading behaviour, and eventually mess up all the picture.[1] In [3] evidence from non linear sigma models was collected which suggested that renormalon growth was not substantially tamed at the eight loop level. In this work not only do we compute on different lattice sizes, but we also push the expansion a couple of orders further, thus imposing a very stringent consistency test. We will give strong evidence that very good control has been obtained over both the IR renormalon growth and the finite size effect on top of that. In particular, the latter turns out to be of the order of a few percent at the order we worked to.

While the eight loop expansion in [1] was a major success, it actually opened more problems than it settled and this sets the stage for the second motivation for our present computation. There was a longstanding problem concerning the possibility of determining the Gluon Condensate (GC) from LGT by a method which actually relied on the computation of high orders in the perturbative expansion of the plaquette. The latter is the lattice representation for the GC and its perturbative expansion is an additive renormalisation which is by far the leading contribution at the value of β usually taken into account. This perturbative contribution can be seen as the contribution coming from the identity operator in the Operator Product Expansion (OPE) for the plaquette. The key point is that in this OPE there is no space for an order a^2 term, since there is no gauge invariant operator of dimension two. The next term in the OPE is expected to be an $O(a^4)$ contribution associated with the genuine GC. The old approach to the problem was to subtract the first (perturbative) contribution from Monte Carlo measurements of the plaquette in order to single out the a^4 term (the GC) [4]. The signature of such a term can be recognized by performing the subtraction at different values of β and looking for the scaling dictated by Asymptotic Freedom. The actual problem is that the series for the plaquette "not only has to be subtracted, it has to be defined in first place".[2] Due to the presence of the IR renormalon, the series has an ambiguity in different prescriptions for resumming it which is just the same order as the GC, i.e. an a^4 contribution. Being aware of this, the most striking point is that by performing the subtraction, with any given prescription, one is left with an a^2 contribution, at least on the 8^4 lattice the computation was first performed on [5]. The problem challenges our understanding of potentially any short-distance expansion and so it needs to be cleared up. At the moment more than one explanation appears to be possible. The

[1] We take once again the chance to thank M. Beneke, who first pointed this out

[2] This is a citation from Beneke's review [2]

effect could either be a pure lattice artifact or a fundamental issue. With respect to the latter hypothesis, [5] suggested the effect of UltraViolet (UV) contributions to the running coupling. Since the time of [5] the phenomenon of non-standard Q^{-2} power effects has actually been reported in other works. In particular, the work in [6] establishes a relation between a dual superconductor confinement mechanism and "OPE-violating" corrections. In this paper we do not want to enter at all the *a^2-affair*. Still, given the potential impact of the result in [5], the reliability of our previous perturbative computations has to be assessed bearing also this question in mind. We will therefore come back to this point when drawing our conclusions. Finally something can be learned from the computation about the NSPT method itself. The present result establishes the highest order ever reached by NSPT in LGT. In [7] a word of caution was spent on the uncritical application of the method, stressing the issue of a careful assessment of statistical errors. Ironically enough, it was pointed out that problems plague simple models much more than complex systems like Lattice SU(3). We refer the reader to a future publication [8] for a more organic report on the application of the NSPT method to LGT. Still, it is reassuring (although of course admittedly not a proof of reliability of the method) to note that in the context of this work there is a fine agreement between the results obtained by our stochastic method and those inferred by a (maybe strong) theoretical prejudice.

The paper is organized as follows. In section 2 we recall the basics of IR Renormalon analysis applied to a dim = 4 condensate and write the formulas relevant for a finite lattice, which are the foundations of our subsequent analysis. In section 3 we present our results for the first ten coefficients in the expansion of the plaquette both on a 8^4 and on a 24^4 lattice, showing that they are consistent with the Renormalon factorial growth and with the expected finite size effects on top of that. Section 4 contains our conclusions and perspectives for future work. An appendix covers algebraic details.

2. The IR renormalon on a finite lattice

We start by writing first of all the definition of our observable, that is the basic plaquette normalised as

$$W(M) \;=\; 1 - \frac{1}{3}\,\langle \mathrm{Tr}\; U_p \rangle \;, \tag{2.1}$$

U_p being the product of links around a 1×1 Wilson Loop. We write an explicit dependence on the size (M) of the lattice on which the observable is computed. In Perturbation Theory

$$W(M) = \sum_{\ell=1} c_\ell^{(M)} \beta^{-\ell} \,. \tag{2.2}$$

Next we recall the main points of Renormalon analysis. Notations are slightly different from those of [1, 5] and closer to those of [3]. The aim of this section is in any case to be self-contained.

2.1 Basics on the factorial growth of perturbative coefficients

The expected form for a dimension 4, Renormalisation Group invariant condensate is written as

$$W = \int_0^{Q^2} \frac{k^2\, dk^2}{Q^4}\, f(k^2/\Lambda^2)\,. \tag{2.3}$$

Q is in our case the UV cutoff fixed by the lattice spacing: $Q = \pi/a$. Given the above dimensional and R.G. arguments, $f(k^2/\Lambda^2)$ is a dimensionless function independent of the scale Q, for large Q, and can thus be expressed in terms of a running coupling at the scale k^2. One obtains the Renormalon contribution by considering the high frequency contribution to eq. (2.3), that is

$$W^{\rm ren} = C \int_{r\Lambda^2}^{Q^2} \frac{k^2\, dk^2}{Q^4}\, \alpha_{\sf s}(k^2)\,, \tag{2.4}$$

in which $f(k^2/\Lambda^2)$ has been taken proportional to the perturbative running coupling (higher powers of the coupling simply result in subleading corrections to the formulas we will get this way). We now introduce the variable

$$z \equiv z_0 \left(1 - \alpha_{\sf s}(Q^2)/\alpha_{\sf s}(k^2)\right), \qquad z_0 \equiv \frac{1}{3b_0}\,, \tag{2.5}$$

using which together with the two loop form for $\alpha_{\sf s}(k^2)$ results in

$$W^{\rm ren} = \mathcal{N} \int_0^{z_{0-}} dz\; e^{-\beta z}\; (z_0 - z)^{-1-\gamma}\,. \tag{2.6}$$

In the last equation we have traded $\alpha_{\sf s}(k^2)$ for the β coupling one is more familiar with on the lattice and introduced a couple of new symbols according to

$$4\pi\alpha_{\sf s}(Q^2) \equiv 6/\beta\,, \qquad \gamma \equiv 2\frac{b_1}{b_0^2}\,, \qquad 0 < z < z_{0-} \equiv z_0(1 - \alpha_{\sf s}(Q^2)/\alpha_{\sf s}(r\Lambda^2))\,. \tag{2.7}$$

In the above equations b_0 and b_1 are the first and second coefficients of the perturbative β-function (to fix normalisation: $b_0 = 11/(4\pi)^4$). z_{0-} is clearly reminiscent of the IR cutoff $r\Lambda^2$ imposed in eq. (2.4) to avoid the Landau pole once the perturbative coupling is plugged in. From eq. (2.6) it is now easy to obtain a perturbative expansion

$$W^{\rm ren} = \sum_{\ell=1} \beta^{-\ell}\, \{c_\ell^{\rm ren} + \mathcal{O}(e^{-z_0\beta})\}\,, \qquad c_\ell^{\rm ren} = \mathcal{N}'\, \Gamma(\ell+\gamma)\, z_0^{-\ell}\,. \tag{2.8}$$

JHEP10(2001)038

The factorial growth of the coefficients of the series is the Renormalon contribution, while the $\mathcal{O}(e^{-z_0\beta})$ is again coming from the IR cutoff $r\Lambda^2$ in eq. (2.4). Naively one could have not imposed any IR cutoff before going from the ill-defined (because of the Landau pole) integral to the series. The factorial growth would have anyway required the $\mathcal{O}(e^{-z_0\beta})$ contribution in order to obtain a resummation of the series. Actually the reader familiar with the subject has most probably already recognized in eq. (2.6) a Borel representation.

2.2 Factorial growth on a finite lattice

Apart from the explicit reference to the lattice spacing in the UV cutoff we have till now adhered to a continuum notation, implicitly assuming some continuum scheme in which for example eq. (2.3) is defined. We now move to rewrite such a representation on a finite lattice. One can consider (notice that we now explicitly write the dependence on the number of lattice points M)

$$W^{\mathrm{ren}}(M) = C \int_{Q_0^2(M)}^{Q^2} \frac{k^2\, dk^2}{Q^4}\, \alpha_{\mathsf{s}}(sk^2)\,. \tag{2.9}$$

The lower limit of integration is the explicit IR cutoff imposed by the finite extent of the lattice ($Q_0(M) = 2\pi/Ma$), while the scale s governs matching from a continuum to a lattice scheme. In view of that and since rescaling $k^2 \to sk^2$ amounts to rescaling $\Lambda \to s^{-1/2}\Lambda$, we expect for s a value $s = K^{-2}$ consistent (up to an eventual further change of scale) with $\Lambda_{\mathrm{cont}} = K\Lambda_{\mathrm{latt}}$ for some continuum scheme. By a change of variable in the integral it is easy to trade the rescaling of the argument of α_{s} for a rescaling of $Q^2 \to sQ^2$ (and similarly for the lower limit of integration). Remember that in the previous section we obtained an expansion in the coupling defined at the scale Q: at one loop level a change of scale in the coupling is equivalent to a change of scheme, and this matches the present notation to the notation of [1, 5].[3] Given eq. (2.9), one can of course perform the same change of variable defined in eq. (2.5) and obtain a new power expansion. In order not to have the main message of the procedure obscured by trivial algebra, we collect the details in an appendix. One is left with

$$W^{\mathrm{ren}}(M) = \sum_{\ell=1} \beta^{-\ell}\, c_\ell^{\mathrm{ren}}(M; s, C)\,. \tag{2.10}$$

The coefficients $c_\ell^{\mathrm{ren}}(M; s, C)$ can be expressed in terms of incomplete Γ functions (see the appendix). It is important to notice their dependences. They depend trivially (i.e. multiplicatively) on the overall constant C which is present in eq. (2.9) to account for everything which is subleading. They also depend on the scale s which governs the matching to the scheme in which eq. (2.4) best describes the expansion (remember that one does not know that *a priori*). Finally, they depend parametrically on the

[3] For further details see later and the appendix.

JHEP10(2001)038

lattice size M. The attitude to take with respect to eq. (2.10) is now the same as in [1, 5], that is: can one determine values $C = \bar{C}$ and $s = \bar{s}$ by fitting the $c_\ell^{\rm ren}(M; s, C)$ to the high orders $c_\ell^{(M)}$ (i.e. the coefficients actually computed on a given lattice size M)? A positive answer states that the perturbative expansion of the plaquette is actually described by the asymptotic dominance of the leading IR Renormalon. Needless to say, once the fit has been done and values $\bar{s}$ and $\bar{C}$ have been obtained, the $c_\ell^{\rm ren}(M; \bar{s}, \bar{C})$ contain in a sense all the information needed to test the IR Renormalon dominance. Given the data at hand (coefficients $c_\ell^{(M)}$ for $\ell \leq \ell_{max} = 10$ and $M = 8, 24$), one can perform a variety of checks. For example, one can fit $C = \bar{C}$ and $s = \bar{s}$ from data up to $\ell < \ell_{max}$ and then compare results at higher loops. As for finite size effects, one can fit the relevant parameters on a given lattice size M (remember that the dependence on M is a parametric one) and then compare $c_\ell^{\rm ren}(M' \neq M; \bar{s}, \bar{C})$ with the actual $c_\ell^{(M')}$. Such comparisons are what we actually proceed to do in the next section.

Before moving to the actual results, we think it is worthwhile to recall something about the crucial issue of finite size effects with respect to the dependence on both M and s of our formulae. The most straightforward way to obtain the IR Renormalon factorial growth goes through the most direct exploitation of eq. (2.4): one simply insert for $\alpha_{\mathsf{s}}(k^2)$ the leading logs (one loop) formula. From this simple recipe one obtains the Γ functions which are the signature of Renormalons. For a dimension 4 condensate (such as the observable at hand) order l turns out to be proportional to (we write the explicit dependence on the IR cut-off)

$$\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \frac{k^4}{Q^4} \ln^{\ell-1}\left(\frac{Q^2}{k^2}\right). \tag{2.11}$$

At this point a simple steepest descent argument points out that momenta k^* that contribute the most to the ℓth order decrease exponentially with ℓ [3]

$$k^* \sim e^{-(\ell-1)/2} \tag{2.12}$$

and that's the reason for being *a priori* worried for a substantial taming of the factorial growth on a finite lattice: starting from a certain order momenta k^* will fall outside the integration region. Now the point is that our formulae actually ask for the scale s in front of the k^2 in the argument of the logarithm which results in

$$k^* \sim s^{-1} e^{-(\ell-1)/2}\,. \tag{2.13}$$

Since $s \ll 1$ this considerably changes the estimate of the dependence on M of relatively high orders in Perturbation Theory. We will see that finite size effects are well in accord with our formulas and under control at the order we got.

n	$c_n^{(8)}$	$c_n^{(24)}$
1	$1.9994(6) \cdot 10^0$	$2.0000(4) \cdot 10^0$
2	$1.2206(16) \cdot 10^0$	$1.2208(10) \cdot 10^0$
3	$2.9523(58) \cdot 10^0$	$2.9621(48) \cdot 10^0$
4	$9.345(27) \cdot 10^0$	$9.417(29) \cdot 10^0$
5	$3.397(14) \cdot 10^1$	$3.439(20) \cdot 10^1$
6	$1.346(7) \cdot 10^2$	$1.368(9) \cdot 10^2$
7	$5.653(34) \cdot 10^2$	$5.774(42) \cdot 10^2$
8	$2.480(18) \cdot 10^3$	$2.545(21) \cdot 10^3$
9	$1.124(10) \cdot 10^4$	$1.159(10) \cdot 10^4$
10	$5.227(52) \cdot 10^4$	$5.416(57) \cdot 10^4$

Table 1: The first ten coefficients in the perturbative expansion of the Lattice SU(3) plaquette.

JHEP10(2001)038

3. The lattice SU(3) plaquette to tenth order

In table 1 we quote the coefficients of the expansion of the SU(3) plaquette to tenth order on 8^4 and 24^4 lattices according to the notation of eq. (2.2). The results were obtained on the *APE*–systems the Milan–Parma group were endowed with by *INFN*. Reaching tenth order (up to the errors one can inspect from the table) required for the 24^4 lattice about 2000 hours on a *qh1* system (128 nodes, delivering a peak performance of 6.4 Gflops). We stress that this system has got an expanded–memory; the 4 MWords on each node allow space enough for a fairly big lattice to such an high order (we recall that in NSPT each field is replicated a number of time which is twice the order in α_s, i.e. the order in g). Results for the 8^4 lattice came out of about 4000 hours on a *q4* system (32 nodes arranged in 4 boards, on each of which a replica of the lattice was allocated; this system delivers a peak performance of 1.6 Gflops). In considering these data and their statistical significance one should keep in mind that the signal to noise ratio is better for the bigger volume. By inspecting the table one can make some first comments. First of all, we improved the errors with respect to [1]. To improve errors on low loops would have been relatively easy, but since we were mainly interested in high loops, the simulation times (and therefore the statistical errors) were actually driven by the tenth order.[4] Having said that, we decided we had the statistical significance we needed when finite size effects were disentangled from statistical errors. By inspecting the coefficients one can see that finite size effects between 8^4 and 24^4 are less than 1% for the first four orders, less than 2% for the first six, less than 3% for the first eight and less than 4% up to the highest order we got, i.e. tenth order. A proper infinite volume extrapolation is

[4] The time spent on a single iteration basically scales as $(O^2 - O)/2$, O being the order in $\beta^{-1/2}$. For more details on technical points concerning the application of NSPT to LGT see [8].

	$M = 8$	δ_8	$M = 24$	δ_{24}	δ_∞
$c_6^{\rm ren}$	$1.30 \cdot 10^2$	-3%	$1.33 \cdot 10^2$	-3%	$< 0.1\%$
$c_7^{\rm ren}$	$5.71 \cdot 10^2$	$+1\%$	$5.84 \cdot 10^2$	$+1\%$	-0.1%
$c_8^{\rm ren}$	$2.51 \cdot 10^3$	$+1\%$	$2.58 \cdot 10^3$	$+2\%$	-0.1%
$c_9^{\rm ren}$	$1.11 \cdot 10^4$	-1%	$1.15 \cdot 10^4$	-1%	-0.2%
$c_{10}^{\rm ren}$	$4.90 \cdot 10^4$	-7%	$5.13 \cdot 10^4$	-6%	-0.3%

Table 2: Perturbative coefficients $c_\ell^{\rm ren}(M; s, C)$ as inferred from **Fit1**. δ_M are per cent deviations from the actual values of $c_\ell^{(M)}$. δ_∞ are the residual per cent deviations of 24^4 coefficients with respect to infinite volume.

out of reach at this level, since one would need better statistics and of course more lattice sizes. Still, we will see that within the IR Renormalon dominance approach it is easy to estimate the correct order of magnitude of such an extrapolation. As we will see, up to tenth order one can infer a correction of the 24^4 results with respect to infinite volume of less than 0.5%.

Having in mind all the points we have already made in the previous sections, we now approach the two basic questions that set the stage for this work, that is: Was the IR Renormalon growth correctly singled out in [1, 5]? Are finite size effects under control? We will show that there is a positive answer for both questions.

3.1 The leading renormalon behaviour

In order to test the IR Renormalon dominance claimed in [1, 5], we now proceed to perform some consistency checks, of just the type we have already referred to in section (2.2). These come out of three different fits which we now describe and which are in a sense a summary of a variety of checks we performed on our results.

- First of all we try to understand to what extent the ninth and tenth order (new) results could be inferred from the first eight (already computed) coefficients. In order to do that, we fit the formula for $c_\ell^{\rm ren}(M; s, C)$ to the set $\{c_7^{(8)}, c_7^{(24)}, c_8^{(8)}, c_8^{(24)}\}$ (notice that while we want to make contact with previous results, nevertheless we now take into account both lattice sizes). We will refer to this procedure as **Fit 1**. As the result of this fit (and later of the following fits) we quote $s_1 = \sqrt{s}$ instead of s. This quantity is simply related to the matching between the Λ parameters between the continuum and lattice scheme one is quite familiar with

$$s_1 = \sqrt{s} = x\,\frac{\Lambda_{\rm latt}}{\Lambda_{\rm cont}}\,. \tag{3.1}$$

  In the previous formula the factor x is a possible further change of scale (see later). We obtain a value for the the scale $s_1 = 0.0074$. In previous works by

JHEP10(2001)038

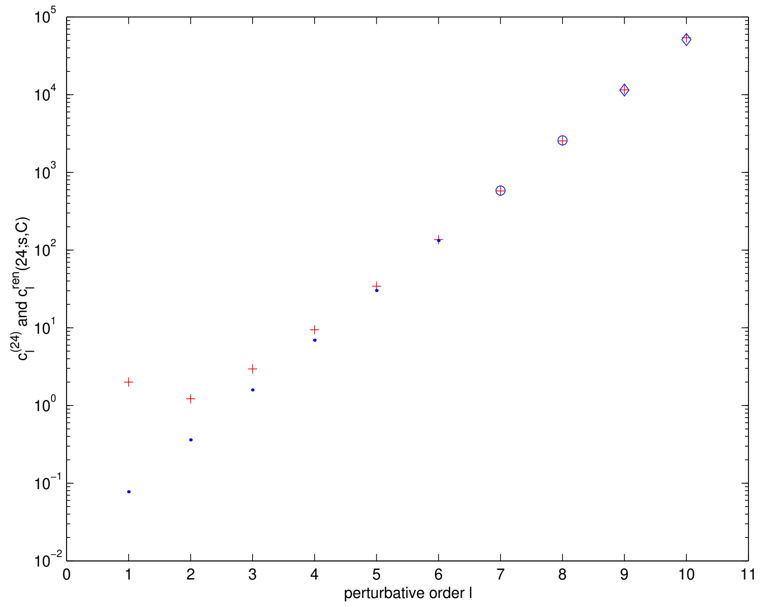

Figure 1: The comparison between $c_\ell^{(24)}$ (red crosses) and $c_\ell^{\rm ren}(24; s, C)$ as inferred from **Fit1**. The latter are marked as circles for the orders that have been taken into account in the fit; they are instead marked as dots or diamonds for other points. In particular diamonds can be looked at as forecasts for the two highest orders computed.

our group the fits were performed without referring to a change of scale in the integral. The matching from a continuum to the lattice scheme was obtained by a change of variable $\beta_{\rm cont} = \beta_{\rm latt} - r - \frac{r'}{\beta_{\rm latt}}$. In terms of the notation of this work the result of the fit in [5] reads $s_1 = 0.0078$. The 5% difference sets a rough order of magnitude for what one has to live with in this and the other comparisons that we make: one should always keep in mind that we are managing notations that differ by higher orders. Notice that this agreement is the first good news for our understanding of finite size effects based on our formula for $c_\ell^{\rm ren}(M; s, C)$: since we now fit results both on the 8^4 and on the 24^4 lattices (while the results in [5] were based on 8^4 alone) the parametric dependence on the lattice size M is supposed to work pretty well (anyway see later for more definite statements). With respect to eq. (3.1) Beneke [2] made the point that the result $s_1 = 0.0078$ in [5] could for example be interpreted by matching to the $\overline{\rm MS}$ scheme and taking into account a further change of scale (the factor x in eq. (3.1)). The latter amounts to computing the $\overline{\rm MS}$ coupling at a scale $0.706/a$ instead of π/a. With the second (standard) choice of scale one obtains the standard result $\Lambda_{\overline{\rm MS}} = 28.8\,\Lambda_{\rm latt}$. $s_1 = 0.0078$ could in turn be interpreted as $0.706/\pi \times 1/28.8 = 0.0078$, while the result $s_1 = 0.0074$ could be read as $0.666/\pi \times 1/28.8 = 0.0074$. One should not take all these considerations too seriously, e.g. there is no compelling commitment to the $\overline{\rm MS}$

scheme (once again, remember that one does not know which is the scheme in which our theoretical prejudice works the best). Still, it is reassuring that the numbers one gets are absolutely sensible so far as their order of magnitude is concerned. Having made contact with our previous results, we now proceed to make contact with what we are mainly interested in, that is the new (ninth and tenth) orders. One can get a first glance at this by looking at figure 1 in which we compare the computed $c_\ell^{(24)}$ with the $c_\ell^{\mathrm{ren}}(24; 0.0074^2, C^{(1)})$ (we will not quote the values obtained for the overall constant C since they are not too enlightening). Notice that different symbols refer to different orders as far as the $c_\ell^{\mathrm{ren}}(24; 0.0074^2, C^{(1)})$ are concerned: we want to emphasize the difference between the orders that are actually included in the fit and those that are regarded as *forecasts* to assess how well the asymptotic behaviour has been singled out. A very good agreement is manifest, which is actually magnified by the logarithmic scale. The numbers themselves are presented in table 2 in which we quote the values of the last five coefficients as inferred from **Fit 1** on both lattice sizes. The first point to be made has to do with finite size effects. As we have already mentioned, a fit to data on both sizes is absolutely sensible and the deviations of the inferred coefficients from the actual ones are much the same on 8^4 and on 24^4. Notice also that the residual finite size effects between 24^4 and infinite volume turn out to be negligible. As for the crucial issue of how well **Fit 1** can single out the asymptotic behaviour we notice that the ninth coefficient is forecast with a precision of 1%, while a 6–7% deviation is there for the tenth order. Notice also that, while orders seven and eight are taken into account in the fit, the agreement for orders less than seven results in a sensible shape for an asymptotic behaviour (by the way, the results are pretty stable to the inclusion of order six in the fit). This is the right time to enphasise a statement that we have already made, i.e. that in inspecting the forecasts of our fits one should keep in mind that we are managing expressions that are there up to higher orders. This is of course not the end of the story. An equally important source of uncertainty is coming from the fact that only *a posteriori* can one understand *how asymptotic* the expansion was at the highest order which has been taken into account in the fit. Roughly speaking, by fitting the expansion for only one quantity there is no obvious way to prevent the fit from pretending to have already reached the actual asymptotic regime.

- With respect to the last point we made there is a nice way to improve. We exploit it in what we call **Fit 2**. This time we make use of the fact that in [1] results for the first eight orders were computed not only for the 1×1 Wilson loop (the basic plaquette), but also for the 2×2.

  Now, the IR Renormalon dominance is supposed to force a *universal* asymptotic behaviour. That's why we now fit the formula for $c_\ell^{\mathrm{ren}}(M; s, C^{(i)})$ to the set

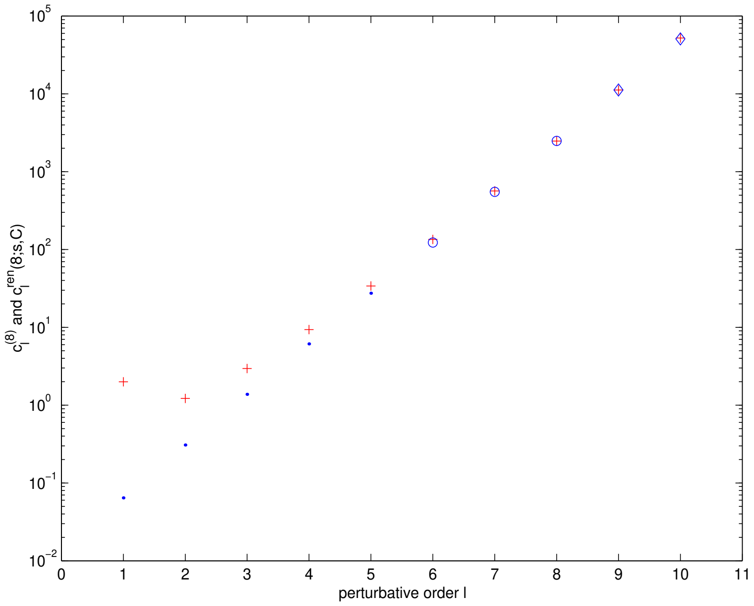

Figure 2: The comparison between $c_\ell^{(8)}$ (red crosses) and $c_\ell^{\rm ren}(8;s,C)$ as inferred from **Fit2**. Same notations as in figure 1.

	$M=8$	δ_8	$M=24$	δ_{24}	δ_∞
$c_6^{\rm ren}$	$1.23\cdot10^2$	-10%	$1.25\cdot10^2$	-10%	$<0.1\%$
$c_7^{\rm ren}$	$5.51\cdot10^2$	-2%	$5.64\cdot10^2$	-2%	-0.1%
$c_8^{\rm ren}$	$2.48\cdot10^3$	$+0.2\%$	$2.55\cdot10^3$	$+0.3\%$	-0.1%
$c_9^{\rm ren}$	$1.12\cdot10^4$	-0.2%	$1.16\cdot10^4$	$+0.2\%$	-0.2%
$c_{10}^{\rm ren}$	$5.09\cdot10^4$	-3%	$5.31\cdot10^4$	-2%	-0.3%

Table 3: Perturbative coefficients $c_\ell^{\rm ren}(M;s,C)$ as inferred from **Fit2**. Same notations as in table 2.

$\{c_6^{(8)},c_7^{(8)},c_8^{(8)},c_6^{(8,2\times2)},c_7^{(8,2\times2)},c_8^{(8,2\times2)}\}$. The $c_\ell^{(8,2\times2)}$ are taken from [1], while of course this time a couple of different overall constant $C^{(i)}$ have to be fitted, corresponding to $i=1\times1$ and $i=2\times2$. Notice that this time, as an extra test of the control over finite size effects, we make use only of 8^4 results for the basic plaquette. The value fitted for the scale is now $s_1=0.0065$. While we are still stopping at eighth order in what we take into account for the fit, the curvature in the growth of the 2×2 coefficients is opposite to that of 1×1 and this corrects a bit with respect to **Fit 1**. As for the 12% change in the scale, much the same that has already been told holds: while there is in a sense extra information on what has to be understood as asymptotic, it is nevertheless reassuring that the change in the coefficients is quite smooth and absolutely sensible as we are managing asymptotic behaviours. Not surprisingly, one can inspect from figure 2 (this time we plot the comparison for coefficients on the 8^4 lattice) and (even better) from table 3 that we have fairly improved the

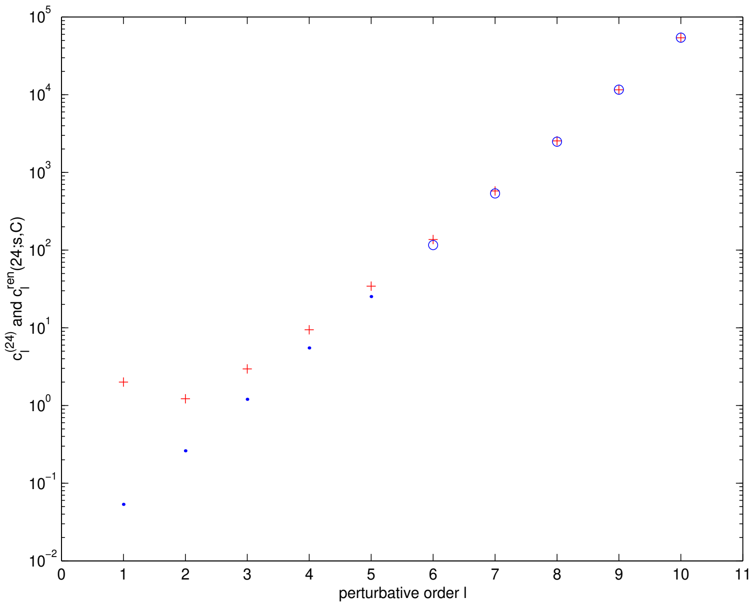

Figure 3: The comparison between $c_\ell^{(24)}$ (red crosses) and $c_\ell^{\rm ren}(24; s, C)$ as inferred from **Fit3**. Same notations as in figure 1. Notice that this time there are no diamonds.

	$M = 8$	δ_8	$M = 24$	δ_{24}	δ_∞
$c_6^{\rm ren}$	$1.14 \cdot 10^2$	-17%	$1.16 \cdot 10^2$	-17%	$< 0.1\%$
$c_7^{\rm ren}$	$5.26 \cdot 10^2$	-7%	$5.37 \cdot 10^2$	-7%	-0.1%
$c_8^{\rm ren}$	$2.43 \cdot 10^3$	-2%	$2.49 \cdot 10^3$	-2%	-0.1%
$c_9^{\rm ren}$	$1.12 \cdot 10^4$	$+0.2\%$	$1.16 \cdot 10^4$	$+0.4\%$	-0.2%
$c_{10}^{\rm ren}$	$5.21 \cdot 10^4$	$< 0.1\%$	$5.43 \cdot 10^4$	$+0.5\%$	-0.3%

Table 4: Perturbative coefficients $c_\ell^{\rm ren}(M; s, C)$ as inferred from **Fit3**. Same notations as in table 2.

asymptotic behaviour. In particular, deviations for the forecasts on order ten are reduced to 1–2%, while other nice features like the consistency of finite size effects are still there. Also the dependence on residual finite size effects turns out to be just the same (negligible) order.

- One can of course devise a fit in which all the information at hand is taken into account. This is what we proceed to do in what we refer to as **Fit 3**. This time we fit the formula for $c_\ell^{\rm ren}(M; s, C^{(i)})$ to the set $\{c_6^{(8)}, c_7^{(8)}, c_8^{(8)}, c_9^{(8)}, c_{10}^{(8)}, c_6^{(24)}, c_7^{(24)}, c_8^{(24)}, c_9^{(24)}, c_{10}^{(24)}, c_6^{(8,2\times2)}, c_7^{(8,2\times2)}, c_8^{(8,2\times2)}\}$.

  We are again changing the players on the ground by including a wide range of orders on both lattice sizes, together with information from the 2×2 Wilson Loop as well. The resulting scale is now $s_1 = 0.0056$, with a change with respect to **Fit 2** which is again the same order of magnitude as that obtained in going from **Fit 1** to **Fit 2**. The result is remarkably stable with respect to keeping into account only the last three orders for the basic plaquette (as for the 2×2 Wilson Loop one can at the same time keep into account only the last order available, that is order eight). Results of the fit are plotted in figure 3 for the 24^4 lattice and summarized in table 4. Of course, this time it does not make any sense to compare the accuracy with which high orders are described, with **Fit 1** and **Fit 2**

In the end, what one can state is that going from **Fit 1** to **Fit 3** one is not going to jeopardize the overall picture: the transition in describing the growth of the coefficients is quite smooth, consistent with the asymptotic behaviour being better and better described. We can give our final estimate for the scale s as resulting from **Fit 3**, also quoting the statistical error-bar deduced from the fit: $s_1 = 0.0056(3)$. As for the $C^{(i)}$ (which we have not yet explicitly referred to till now) one can inspect slightly bigger error-bars: $C^{(1\times1)} = 0.53(4)$, $C^{(2\times2)} = 0.43(4)$. Again, we stress that these values were found to be valid at any volume, thus validating the finite size effects implicit in our formulae. Still, in the next section we give a more quantitative estimate of the finite size effects.

JHEP10(2001)038

3.2 Finite size effects

From all the previous arguments the impact of finite size effects has already been widely discussed. We have already shown in **Fit 2** how one can obtain a good control on asymptotics by taking into account in the fit only the smaller lattice size. To give a final assessment of the impact of finite size effects we now proceed to compare our final results for $(s, C^{(i)})$ emerging from **Fit 3** with the ones we get by taking into account the set $\{c_6^{(8)}, c_7^{(8)}, c_8^{(8)}, c_9^{(8)}, c_{10}^{(8)}, c_6^{(8,2\times2)}, c_7^{(8,2\times2)}, c_8^{(8,2\times2)}\}$ (that is, all the orders, but only one lattice size). In the latter case we obtain $s_1 = 0.0057(4)$, $C^{(1\times1)} = 0.54(5)$, $C^{(2\times2)} = 0.43(4)$. As one can see, the systematic effects of finite size are dominated by statistical errors. We stress for the last time that this is a proof that finite size effects are not going to mess the renormalon picture up: they are well under control while extracting asymptotics. At this point we think it is worthwhile to present the tests of finite size effects also in a graphical format. In figure 4 we plot the ratios $c_\ell^{(8)}/c_\ell^{\mathrm{ren}}(8; 0.0065^2, C^{(1\times1)})$ and $c_\ell^{(24)}/c_\ell^{\mathrm{ren}}(24; 0.0065^2, C^{(1\times1)})$ as deduced from **Fit 2**.

The deviation from unity of these ratios tell us how much the measured coefficients deviate from their asymptotic behaviour. Since in **Fit 2** only the 8^4 lattice has been taken into account for the fit (and only to the order β^{-8}), one expects some deviation from unity for the $c_\ell^{(24)}$ (and for the highest order). In figure 5 the only difference is that the highest orders have been also included in the fit (but only from the 8^4 lattice). While only the 8^4 lattice has been taken into account in the fit, the points for both lattices fall on top of each other, i.e. from a fit to 8^4 and the dependence on M embedded in our formulae we also describe the coefficients of the 24^4 lattice with basically the same accuracy.

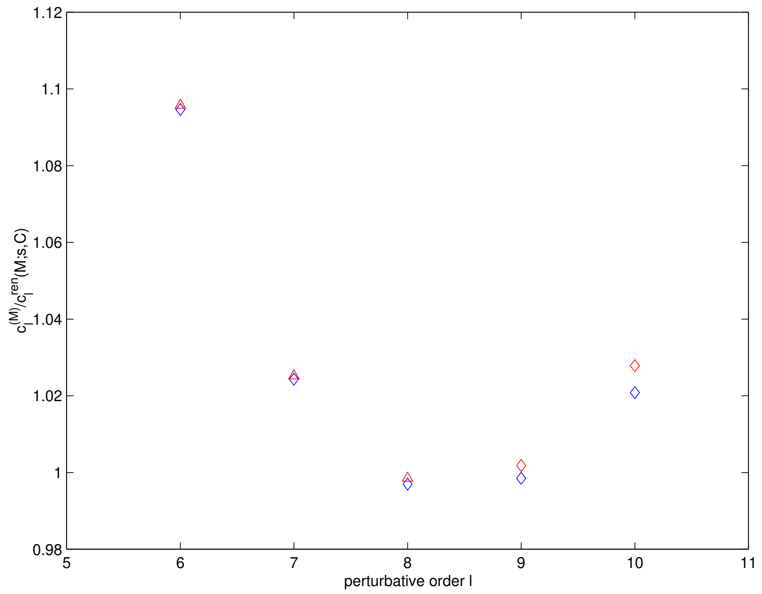

Figure 4: The ratios $c_\ell^{(8)}/c_\ell^{\rm ren}(8;s,C)$ (red triangles pointing up and red diamonds) and $c_\ell^{(24)}/c_\ell^{\rm ren}(24;s,C)$ (blue diamonds) as inferred from the results of **Fit 2**.

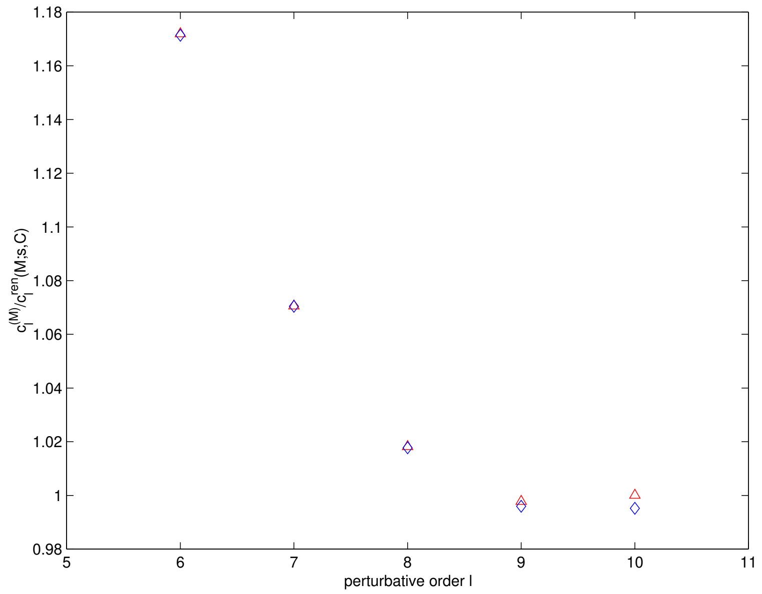

Figure 5: Same quantities as in figure 4 as obtained by the results of the equivalent of **Fit 3** in which only 8^4 lattice has been taken into account. Quantities for 8^4 are plotted as red triangles pointing up while quantities for 24^4 are plotted as blue diamonds.

4. Conclusions and perspectives

We computed the perturbative expansion of the basic SU(3) plaquette on both 8^4 and 24^4 lattices to β^{-10} order. Computing such high orders was possible only by making use of NSPT. One could once again wonder whether it was worth the effort.

We think that NSPT gives in this context a unique chance to validate the conjecture of renormalon dominance and with this respect the present computation fills the only gap left over by our previous perturbative computations in [1, 5], i.e. complete assessment of finite size effects. Our conclusion is that they do not mess the picture up: the coefficients growth is consistent with both the leading IR Renormalon dominance and with finite size effects on top of that. The impact of the latter in comparing results on 8^4 and 24^4 does not exceed 4% at the order we got and the residual finite size effects on the 24^4 results are no more than half a percent, again at tenth order.

We finally come back to a point already made in the introduction, that is what insight one can gain from the present computations with respect to the odd result in [5], i.e. the extraction of a quadratic contribution to the lattice representative for the Gluon Condensate. In this work we checked that finite size effects did not cause a misunderstanding of the renormalon divergence even on a lattice as small as 8^4, being well within the statistical errors. In this respect the new information coming from the present computation only amounts to subleading contributions, like the dependence on the scale s in our formalism or corrections to avoid double counting once one stops the perturbative computations at a given order and then proceeds to resum. In the end, the result of the subtraction performed in [5] is unchanged. This of course does not imply that the odd result of [5] is not a finite size effect itself. This question can only be answered by performing the whole subtraction procedure on a bigger lattice. With this respect, there is therefore new information coming from the present study, since we now control perturbation theory on a fairly large 24^4 lattice. We stated from the beginning that we did not want to enbark on an actual refinement of the subtraction procedure in the present work, leaving it for a subsequent publication [9]. As a matter of fact we think that simply repeating the procedure of [5] on a bigger lattice would not be satisfactory. To better investigate the issue of an a^2 contribution in the subtraction, we think that one should address other relevant issues. A valuable point is a systematic study of different prescriptions to deal with zero-modes, the related impact of toron configurations and (again) finite size effects on top of all that. One should also try to assess the scenario suggested in [2], that is an a^2 contribution could come from subleading corrections to the lattice action with respect to the continuum theory. Finally, a possible connection to the results in [6] could be investigated. All these points are a framework for the subsequent analysis that we plan to perform.

Acknowledgments

The authors are grateful to G. Burgio for many interesting discussions and to E. Onofri and G. Marchesini for their constant interest and encouragement. It was first M. Beneke who urged our group to worry about and to carefully assess finite size effects: we take once again the chance to acknowledge this precious suggestion. F. D.R.

JHEP10(2001)038

acknowledges support from both Italian MURST under contract 9702213582 and from I.N.F.N. under *i.s. PR11*. L.S. acknowledges support from EU Grant EBR-FMRX-CT97-0122.

A. Algebraic details for finite size formulae

We now sketch the steps to go from eq. (2.9) to eq. (2.10). As we have already said in section (2.2) one way to treat the dependence on s is to rescale the integration variable (define $t^2 \equiv sk^2$). In this way one ends up with the rescaling $Q^2 \to sQ^2$. This in turns means $\beta \to \beta(sQ^2)$ which makes contact with the formulation of [1, 5] (again, as already said): $\beta \to \beta - r$ (at this level one obtains a one loop formula). We chose another way to proceed which manipulates the integrand in another way. By going through the change of variable of eq. (2.5) one obtains (up to overall constants)

$$\int_0^{z_{ir}(M)} dz\, e^{-\beta z}\, (z_0 - z)^{-1-\gamma} \frac{1}{1 + 6b_0 \frac{z_0}{z_0 - z} \frac{1}{\beta} \ln s}\,, \tag{A.1}$$

$z_{ir}(M)$ being the value for z pertaining to the IR cut–off $Q_0(M)$. The last factor is simply $\frac{\alpha_s(sk^2)}{\alpha_s(k^2)}$. By expanding the latter in a geometric series (we are aiming at an expansion in β^{-1}) and by performing a change of variable we end up with

$$\beta^{\gamma} e^{-\beta z_0} (-1)^{-1-\gamma} \sum_n (6b_0 z_0 \ln s)^n \int_{-\beta z_0}^{\beta z_{ir}(M) - \beta z_0} e^{-z} z^{-1-\gamma-n} dz\,. \tag{A.2}$$

Note that the effect of the scale s propagates to high orders much the same way the leading logs expansion for α_s is responsible for Renormalon growth, i.e. via a geometric series. The last input is now the asymptotic expansion for incomplete Γ functions (which is useful once one splits the previous integral in terms of a sum of incomplete Γ functions)

$$\int_z^{\infty} dt\, e^{-t}\, t^{a-1} \approx z^{a-1} e^{-z} \sum_{k \geq 0} \frac{\Gamma(k+1-a)}{\Gamma(1-a)} (-z)^{-k}\,. \tag{A.3}$$

Once this expansion (valid for $|z| \to \infty$ in $|\arg z| < 3\pi/2$) is plugged in, one proceeds to the final power expansion in β^{-1} (which is easy to manage for example in *Mathematica*).

References

[1] F. Di Renzo, E. Onofri and G. Marchesini, *Renormalons from eight loop expansion of the gluon condensate in lattice gauge theory*, *Nucl. Phys.* **B 457** (1995) 202 [hep-th/9502095].

[2] For a recent review and a complete listing of classical references see M. Beneke, *Phys. Rev.* **317** (1999) 1

[3] F. Di Renzo, G. Marchesini and E. Onofri, *Infrared renormalons and finite volume*, *Nucl. Phys.* **B 497** (1997) 435 [hep-lat/9612016].

[4] See for example B. Allés, M. Campostrini, A. Feo and H. Panagopoulos, *The three loop lattice free energy*, *Phys. Lett.* **B 324** (1994) 433 [hep-lat/9306001] and references therein.

[5] G. Burgio, F. Di Renzo, G. Marchesini and E. Onofri, Λ^2 *contribution to the condensate in lattice gauge theory*, *Phys. Lett.* **B 422** (1998) 219

[6] M.N. Chernodub, F.V. Gubarev, M.I. Polikarpov and V.I. Zakharov, *Confinement and short distance physics*, *Phys. Lett.* **B 475** (2000) 303 [hep-ph/0003006] and referencs therein

[7] R. Alfieri, F. Di Renzo, E. Onofri and L. Scorzato, *Understanding stochastic perturbation theory: toy models and statistical analysis*, *Nucl. Phys.* **B 578** (2000) 383 [hep-lat/0002018].

[8] F. Di Renzo and L. Scorzato, in preparation.

[9] G. Burgio, F. Di Renzo and L. Scorzato, in preparation.

JHEP10(2001)038